Lecture 7:

Discrete Fourier Transform:

Definition: The 1D discrete Fourier Transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, 2, \ldots, N-1$ is defined as:

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k)\, e^{-j\frac{2\pi m k}{N}} \quad \left(\text{where } j = \sqrt{-1},\ e^{j\theta} = \cos\theta + j\sin\theta \right)$$

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

Image decomposition under DFT:

Consider a $N \times N$ image g, the DFT of g:

$$\hat{g} = \mathcal{U} g \mathcal{U} \quad \text{(DFT in matrix form)}$$

where $\mathcal{U} = (\mathcal{U}_{kl})_{0 \le k, l \le N-1} \in M_{N \times N}$ and $\mathcal{U}_{kl} = \frac{1}{N} e^{-j\frac{2\pi kl}{N}}$.

$$\vec{u}_m = \begin{pmatrix} e^{-j\frac{2\pi(m(0))}{N}} \\ \vdots \\ e^{-j\frac{2\pi(ml)}{N}} \\ \vdots \\ e^{-j\frac{2\pi(m(N-1))}{N}} \end{pmatrix}$$

Note that $\mathcal{U}$ is symmetric and $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^* \mathcal{U}$

$$g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \, \vec{w}_k \vec{w}_l^{\,T}$$

Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$

$$\hat{g}(k,l) = \sum_m \sum_n g(m,n) e^{-j\frac{2\pi km}{N}} e^{-j\frac{2\pi ln}{N}} = \sum_m \sum_n g(m,n) \Big[\underbrace{\begin{pmatrix} e^{-j\frac{2\pi k(0)}{N}} \\ \vdots \\ | \end{pmatrix}}_{\vec{u}_m} \underbrace{\big(\longrightarrow \big)}_{\vec{u}_m^{\,T}} \Big]_{k,l}$$

Remark:

Note that $UU^* = \frac{1}{N} I$. $\therefore$ U is not unitary.

If we normalize U to $\tilde{U} = \sqrt{N} U$. Then $\tilde{U}$ is unitary!

Some other definition of DFT:

(1D) $$\hat{f}(m) = \frac{1}{\sqrt{N}} \sum_{n=0}^{k-1} f(k) e^{-j\left(\frac{2\pi m k}{N}\right)}$$

(2D) $$\hat{f}(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l)\, e^{-j2\pi\left(\frac{mk+nl}{N}\right)}$$

In this case, let $\tilde{U} = (\tilde{U}_{kl})_{0 \le k, l \le N-1}$; $\tilde{U}_{kl} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi k l}{N}}$. Then: $\hat{f} = \tilde{U} f \tilde{U}$

Then, $\tilde{U} = \sqrt{N} U$

$\therefore$ Normalizing the definition of DFT $\Rightarrow$ unitary $\tilde{U}$ can be applied!

BUT: Inverse DFT must be adjusted!!

Why is DFT useful in imaging:

1. DFT of convolution:

Recall: $$g * w(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, w(n', m')$$

$$\left(g, m \in M_{N \times M}(\mathbb{R}) \right)$$

Then, the DFT of $g * w$ = $MN\, DFT(g)\, DFT(w)$

$\therefore$ DFT of convolution can be reduced to simple multiplication!

Recall: Shift-invariant image transformation = 2D convolution.

$\therefore$ Easy computation/manipulation of shift-invariant transf. after DFT!!

Proof:

DFT of $g * w$ at (p, q)

$$= \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g*w(n,m)\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

$$= \frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g(n-n', m-m')\, w(n', m')\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

Change of variables:

$n \rightarrow n'' = n - n'$

$m \rightarrow m'' = m - m'$

$$= \underbrace{\frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} w(n', m')\, e^{-j2\pi\left(\frac{pn'}{N} + \frac{qm'}{M}\right)}}_{\hat{w}(p,q)} \underbrace{\sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{M-1-m'} g(n'', m'')\, e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)}}_{T(p,q)}$$

Note that: g and w are periodically extended.

$\therefore\ g(n-N, m) = g(n,m)$ and $g(n, m-M) = g(n,m)$

$$\therefore\ T \equiv \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=-n'}^{-1} g(n'', m'')\, e^{-j2\pi\frac{pn''}{N}} + \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=0}^{N-1-n'} g(n'', m'')\, e^{-j2\pi\left(\frac{pn''}{N}\right)}$$

Consider $\sum_{n''=-n'}^{-1} g(n'', m'') e^{-j2\pi \frac{pn''}{N}} \overset{n'''=N+n''}{=\!=} \sum_{n'''=N-n'}^{N-1} \underbrace{g(n'''-N, m'')}_{g(n''', m'')} e^{-j2\pi\left(\frac{pn''}{N}\right)} \cancelto{1}{e^{j2\pi p}}$

We can do similar thing for index m''.

$$\therefore\ T = \sum_{m''=0}^{M-1} \sum_{n''=0}^{N-1} g(n'', m'') e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)} = MN\,\hat{g}(p, q)$$

$$\therefore\ \widehat{g * w}\,(p, q) = MN\,\hat{g}(p, q)\,\hat{w}(p, q)$$

Remark: Conversely, if $x(n, m) = g(n, m)\,w(n, m)$

Then, $\hat{x}(k, l) = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \hat{g}(p, q)\,\hat{w}(k-p, l-q)$ (Convolution of g and w)

2. Average value of image

$$\text{Average value of } g = \bar{g} = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l) = \underbrace{\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi(0)}}_{\hat{g}(0,0)}$$

3. DFT of a shifted image

Let $g = (g(k', l'))$ be a $N \times N$ image, where the indices are taken as:

$$-k_0 \leq k' \leq N-1-k_0 \quad \text{and} \quad -l_0 \leq l' \leq N-1-l_0$$

Let $\tilde{g}$ be shifted image of g defined as:

$$\tilde{g}(k,l) = g(k-k_0, l-l_0) \quad \text{where} \quad 0 \leq k \leq N-1, \quad 0 \leq l \leq N-1$$

Then:
$$\hat{\tilde{g}}(m,n) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k-k_0, l-l_0)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

$$= \underbrace{\frac{1}{N^2}\sum_{k'=-k_0}^{N-1-k_0}\sum_{l'=-l_0}^{N-1-l_0} g(k', l')\, e^{-j2\pi\left(\frac{k'm+l'n}{N}\right)}}_{\hat{g}(m,n)}\, e^{-j2\pi\left(\frac{k_0 m + l_0 n}{N}\right)}$$

$$\therefore \quad \hat{g}(m,n) = \hat{g}(m,n)\, e^{-j2\pi\left(\frac{k_0 m + l_0 n}{N}\right)}$$

<u>Remark:</u> $\hat{g}(m-m_0,\, n-n_0) = \mathrm{DFT}\left(g \times e^{j2\pi\left(\frac{m_0 k + n_0 l}{N}\right)}\right)$ with carefully chosen indices!